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THE GENERAL FORM OF THE SMITH-HELMHOLTZ EQUATION

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ABSTRACT. The Smith-Helmholtz equation $\mu \sin \theta y = \mu' \sin \theta' y'$ is only applicable to infinitesimal objects and images. When both field and aperture are large, in addition to the ordinary equation involving lengths measured along a ray, there are three exact equations of the type

$$xx'a + \mu' L' x' b - \mu L x c - \mu L \mu' L' d = 0, \quad bc - ad = 1,$$

all expressing the same correspondence. This correspondence is primarily geometrical rather than optical. The coefficient a in all these equations is the secondary power of the system for the ray considered. When the instrument is axially symmetrical the coefficients in two of the equations have the values $b = c = 1$, $d = 0$. The third equation can be made to take the same form by moving the origins to suitable points on the axis of the instrument. The equations are considered in their application to asymmetrical instruments. In telescopic systems axial symmetry is, in general, necessary for the Smith-Helmholtz equality to be applicable. From the general equations a great variety of formulae for computing conjugate points can be obtained. Equations of the same one-one class can be found for other projective relations, but, in general, they are less useful than the corrected Smith-Helmholtz equations, because when the linear and angular variables are associated as desired the positions of the origins must be computed instead of being recognisable by inspection.

Introduction

ONE of the most interesting chapters in the theory of optical instruments was begun when Robert Smith showed that the relative dimensions y and y' of a linear object placed perpendicular to the axis of a symmetrical optical instrument and its image can be derived from the axial inclinations θ and θ' of the initial and final parts of a ray which passes through the axial point of the object, by means of the equation $\mu\theta y = \mu'\theta' y'$, where μ and μ' are the refractive indices of the extreme media. The great importance of this equation has been recognised ever since its discovery, but with the advent of instruments in which the departures from collinear imagery

were appreciable the law lost some of its old utility, for it only holds for small values of θ and θ'. For large inclinations we now know that the Helmholtz extension $\mu \sin \theta y = \mu' \sin \theta' y'$, or Abbe's equivalent expression for the magnification y'/y, must be used. Lately it has been realised that the form of the relation obtained by substituting for $\sin \theta$ the cosine of the angle between the ray and the y-axis is more fundamental. These equations in their various forms have been the subject of many papers, but as far as I can recollect the suggestion that these familiar forms are degenerate examples of a relation, applicable to large angular fields as well as to large angular apertures, of the same type as the general one-one relation familiar to all students of optics from the equations for calculating conjugate points, has not been made previously. In fact the equations for finding the position of an image may be regarded as special examples of the general relation from which the Smith-Helmholtz equation also is derived as a special case. The fundamental form of the latter equation has been obscured in consequence of an assumption that the lengths y and y' are to be obtained from the traces of rays other than those to which the θ and θ' refer. When this rather peculiar procedure is altered, so that all the quantities entering into the equation depend upon the same ray, we find that the equation in its known form is incomplete, and that Abbe's ratio is formally preferable to Helmholtz' linear equation. By adhering to the complete equation we can construct relations which will hold for any system of reflecting or refracting surfaces and also remove the restriction that the linear magnitudes must be small.

The equations for a single surface

We may conveniently start from the well-known analytical equations for the directions of a ray refracted at a point of a surface where the normal has the direction (l, m, n), viz.

$$\frac{\mu' L' - \mu L}{l} = \frac{\mu' M' - \mu M}{m} = \frac{\mu' N' - \mu N}{n} = \mu' \cos \phi' - \mu \cos \phi,$$

together with $\mu' \sin \phi' = \mu \sin \phi$, where ϕ and ϕ' are the angles made with the normal by the incident ray (L, M, N) and the refracted ray (L', M', N') respectively. We will suppose that an arbitrary point on the normal is taken as origin, and that this lies at a distance r from the point of refraction (ξ, η, ζ). We may re-write the preceding equations in a form involving (ξ, η, ζ) in place of (l, m, n),

$$\frac{\mu' L' - \mu L}{\xi} = \frac{\mu' M' - \mu M}{\eta} = \frac{\mu' N' - \mu N}{\zeta} = -\frac{\mu' \cos \phi' - \mu \cos \phi}{r} \equiv -K \qquad \ldots\ldots(1).$$

Let (x, y, z) and (x', y', z') be points on the incident and emergent rays distant respectively ρ and ρ' from the point of refraction, so that

$$\frac{x - \xi}{L} = \frac{y - \eta}{M} = \frac{z - \zeta}{N} = \rho,$$

$$\frac{x' - \xi}{L'} = \frac{y' - \eta}{M'} = \frac{z' - \zeta}{N'} = \rho'.$$

On replacing the direction cosines in (1) we obtain

$$\frac{1}{\xi}\left(\frac{\mu' x'}{\rho'} - \frac{\mu x}{\rho}\right) = \frac{1}{\eta}\left(\frac{\mu' y'}{\rho'} - \frac{\mu y}{\rho}\right) = \frac{1}{\zeta}\left(\frac{\mu' z'}{\rho'} - \frac{\mu z}{\rho}\right) = \frac{\mu'}{\rho'} - \frac{\mu}{\rho} - K \quad \ldots\ldots(2).$$

In this general form equations (2) are merely the equations of refraction disguised. To obtain relations of value we must make them less general. Let us suppose that the two points (x, y, z) and (x', y', z') lie in a plane, not coincident with the plane of refraction, which passes through the origin. If its direction cosines are (u, v, w) we then have

$$ux + vy + wz = 0,$$
$$ux' + vy' + wz' = 0,$$
$$u\xi + v\eta + w\zeta \neq 0,$$

and therefore from (2)

$$\frac{\mu'}{\rho'} - \frac{\mu}{\rho} - K = 0 \qquad \ldots\ldots(3),$$

and

$$\frac{x'}{x} = \frac{y'}{y} = \frac{z'}{z} = \frac{\rho'}{\mu'} \Big/ \frac{\rho}{\mu} \qquad \ldots\ldots(4).$$

When r is a principal radius of curvature of the surface at (ξ, η, ζ) the origin coincides with the corresponding centre of curvature, and equation (3) becomes the ordinary equation for secondary conjugate points. The derivation shows that it may be interpreted more generally as applying to any three collinear points on the incident ray, the refracted ray, and the normal.

From (4) we have identically

$$\begin{aligned} 0 &= \mu' x' L' . \rho L - \mu x L . \rho' L' \\ &= \mu' x' L' (x - \xi) - \mu x L (x' - \xi) \\ &= xx' (\mu' L' - \mu L) - \xi (\mu' x' L' - \mu x L), \end{aligned}$$

or by (1)

$$\left.\begin{aligned} \mu' x' L' - \mu x L + xx' K &= 0 \\ \mu' y' M' - \mu y M + yy' K &= 0 \\ \mu' z' N' - \mu z N + zz' K &= 0 \end{aligned}\right\} \qquad \ldots\ldots(5),$$

the last two equations following from the symmetry of (1) and (4). It will be observed that these are generalisations of the equations from which we started. For example (1) is the particular case when (x, y, z) and (x', y', z') coincide, that is when they fall on the point of refraction, and the three-point line becomes the normal to the refracting surface.

The importance of geometrical symmetry

Like the equations from which they are derived, equations (5) are primarily of geometrical, not of optical, significance. Their utility in optics depends on our ability to recognise that projection through special points on the normals to the surfaces will, in consequence of peculiarities in the construction of the instrument, be of optical importance. The typical example is projection through the centre of a spherical surface. From symmetry it is evident, without detailed demonstration, that there will be a great concentration of radiant energy at the point of a refracted ray determined from the radiant point by central projection. In more complex

systems symmetry of construction is of great assistance in enabling us to select the significant projection centres.

The equations for several surfaces

When in any instrument, symmetrical or otherwise, a train of significant projection centres has been determined on all the normals corresponding to a known ray, we may deduce equations connecting the initial and final points so related. In general, these equations will be of the form

$$awv' + bv' - cv - d = 0,$$

and on combining any two systems the coefficients of the new equation for the joint instrument may be obtained from the old coefficients by matrix multiplication. For example, if we write w_1, w_2, w_3 for μLx, μMy, μNz respectively in the initial medium, and add accents for the final medium, we have the general equations

$$a_n w_n w_n' + b_n w_n' - c_n w_n - d_n = 0, \qquad (n = 1, 2, 3) \qquad \ldots\ldots(6),$$

where, if the system has two surfaces distinguished by the suffixes 1 and 2, separating media distinguished by the suffixes 0, 1 and 2, the coefficients may be found from

$$\left.\begin{aligned}
\begin{pmatrix} b_1 & d_1 \\ a_1 & c_1 \end{pmatrix} &= \begin{pmatrix} 1 & . \\ \dfrac{K_1}{\mu_0 L_0 . \mu_1 L_1} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\mu_1 L_1 X_1 \\ . & 1 \end{pmatrix} \begin{pmatrix} 1 & . \\ \dfrac{K_2}{\mu_1 L_1 . \mu_2 L_2} & 1 \end{pmatrix} \\
\begin{pmatrix} b_2 & d_2 \\ a_2 & c_2 \end{pmatrix} &= \begin{pmatrix} 1 & . \\ \dfrac{K_1}{\mu_0 M_0 . \mu_1 M_1} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\mu_1 M_1 Y_1 \\ . & 1 \end{pmatrix} \begin{pmatrix} 1 & . \\ \dfrac{K_2}{\mu_1 M_1 . \mu_2 M_2} & 1 \end{pmatrix} \\
\begin{pmatrix} b_3 & d_3 \\ a_3 & c_3 \end{pmatrix} &= \begin{pmatrix} 1 & . \\ \dfrac{K_1}{\mu_0 N_0 . \mu_1 N_1} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\mu_1 N_1 Z_1 \\ . & 1 \end{pmatrix} \begin{pmatrix} 1 & . \\ \dfrac{K_2}{\mu_1 N_1 . \mu_2 N_2} & 1 \end{pmatrix}
\end{aligned}\right\} \ldots\ldots(7),$$

and X_n, Y_n, Z_n are the coordinates of the origin for refraction at the $(n+1)$th surface referred to those for the preceding surface. The extension to more complex systems is obvious.

Canonical forms of the equations

In every set of these coefficients the identity $bc - ad = 1$ will clearly hold. From an inspection of the matrix factors it is obvious that the familiar values $b = c = 1$, $d = 0$ are a direct expression of the fact that all the projective centres are coplanar. When all these centres are collinear the conditions $b = c = 1$, $d = 0$ will hold for two sets of coefficients. The further condition $a = 0$, which is needed to make Helmholtz' equation exact and not merely the limit towards which the relation tends when y and y' are diminished indefinitely, is not, in general, satisfied. In fact it will only be true when the imagery under consideration is telescopic, for the new equations determine the same sets of conjugate points as the more familiar equations obtained by compounding the corresponding set of equations (3). It is simple to derive any one set from another set. For example, if the usual equation is

$$a \frac{\rho}{\mu} \cdot \frac{\rho'}{\mu'} + b \frac{\rho'}{\mu'} - c \frac{\rho}{\mu} - d = 0 \qquad \ldots\ldots(8),$$

where, if the system consists of two surfaces,

$$\begin{pmatrix} b & d \\ a & c \end{pmatrix} = \begin{pmatrix} 1 & . \\ K_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{s_1}{\mu_1} \\ . & 1 \end{pmatrix} \begin{pmatrix} 1 & . \\ K_2 & 1 \end{pmatrix} \qquad \text{.....(9),}$$

s_1 being the length of the path between the surfaces, the x-variables of the extreme media are connected with the ρ's by the equations

$$\xi + \rho L = x = \frac{w_1}{\mu L}, \quad \xi' + \rho' L' = x' = \frac{w_1'}{\mu' L'}.$$

Substituting these in (8) and rearranging we have

$$\frac{a w_1 w_1'}{\mu L . \mu' L'} + \frac{(\mu L b - a\xi)\, w_1'}{\mu' L'} - \frac{(\mu' L' c + a\xi')\, w_1}{\mu L} - \{\mu L . \mu' L' d - \mu' L' \xi c + \mu L \xi' b - \xi\xi' a\} = 0.$$

Since $bc - ad = b_1 c_1 - a_1 d_1 = 1$, we see that this equation is identical with

$$a_1 w_1 w_1' + b_1 w_1' - c_1 w_1 - d_1 = 0.$$

It follows that

$$\left.\begin{gathered} a_1 \mu L \mu' L' = a_2 \mu M \mu' M' = a_3 \mu N \mu' N' = a, \\ \frac{b_1}{a_1} = \mu L \left(\frac{b}{a}\mu L - \xi\right), \quad \frac{b_2}{a_2} = \mu M \left(\frac{b}{a}\mu M - \eta\right), \quad \frac{b_3}{a_3} = \mu N \left(\frac{b}{a}\mu N - \zeta\right) \\ \frac{c_1}{a_1} = \mu' L' \left(\frac{c}{a}\mu' L' + \xi'\right), \quad \frac{c_2}{a_2} = \mu' M' \left(\frac{c}{a}\mu' M' + \eta'\right), \quad \frac{c_3}{a_3} = \mu' N' \left(\frac{c}{a}\mu' N' + \zeta'\right) \end{gathered}\right\} \text{(10),}$$

the equations of the last two lines being generalisations of (1), which applies only to a single surface. They illustrate the great importance of the ratios obtained by dividing any b or c by the corresponding a. The relations of chief interest for our present purposes are those appearing in the first line. They show that in the standard form the equations will all have the power a as the coefficient of product terms in $\rho\rho'/\mu\mu'$, xx', yy' and zz' respectively. When the projective centres are coplanar we always have one equation identical in form with one of equations (5), and when the centres are collinear two equations are of this form. In these classes the remaining equations, and in other systems all the equations, take the more general forms, such as

$$axx' + \mu' L' b_1 x' - \mu L c_1 x - \mu L \mu' L' d_1' = 0,$$

where $\mu L \mu' L' d_1' = d_1$, $b_1 c_1 - a d_1' = 1$, and, in general, $b_1 \neq c_1$, $d_1' \neq 0$. They may be transformed in various ways corresponding to the usual treatment of the paraxial relation for conjugate points. Thus

$$(ax + \mu' L' b_1)(ax' - \mu L c_1) + \mu L \mu' L' = 0$$

is the last equation expressed in Newtonian form. By changing the object space origin to the point whose coordinates referred to the first surface projection centre are

$$\left(\frac{\mu' L'(1 - b_1)}{a}, \quad \frac{\mu' M'(1 - b_2)}{a}, \quad \frac{\mu' N'(1 - b_3)}{a}\right) \text{ or } \left(\frac{1 - b_1}{\mu L a_1}, \quad \frac{1 - b_2}{\mu M a_2}, \quad \frac{1 - b_3}{\mu N a_3}\right),$$

and similarly the image space origin to the point whose coordinates referred to the last surface projection centre are

$$\left(\frac{\mu L(c_1-1)}{a}, \quad \frac{\mu M(c_2-1)}{a}, \quad \frac{\mu N(c_3-1)}{a}\right) \text{ or } \left(\frac{c_1-1}{\mu' L' a_1}, \quad \frac{c_2-1}{\mu' M' a_2}, \quad \frac{c_3-1}{\mu' N' a_3}\right),$$

the equations become

$$\begin{aligned} axx' + \mu' L' x' - \mu L x &= 0, \\ ayy' + \mu' M' y' - \mu M y &= 0, \\ azz' + \mu' N' z' - \mu N z &= 0, \end{aligned}$$

which may be regarded as the canonical forms of the Smith-Helmholtz equation for non-telescopic systems. This simplification corresponds to the reduction of (8) to the canonical form

$$a \cdot \frac{\rho}{\mu} \cdot \frac{\rho'}{\mu'} + \frac{\rho'}{\mu'} - \frac{\rho}{\mu} = 0,$$

by moving the two origins along the ray from the first and last points of refraction to the positions

$$\rho_0 = \mu \frac{1-b}{a}, \quad \rho_0' = \mu' \frac{c-1}{a}$$

respectively. Referred to the original projective centres these new origins on the ray are at $(\xi + \rho_0 L, \eta + \rho_0 M, \zeta + \rho_0 N)$ and $(\xi' + \rho_0' L', \eta' + \rho_0' M', \zeta' + \rho_0' N')$, or by (10) at

$$\left(\frac{\mu L - \mu' L' b_1}{a}, \quad \frac{\mu M - \mu' M' b_2}{a}, \quad \frac{\mu N - \mu' N' b_3}{a}\right)$$

and

$$\left(\frac{\mu L c_1 - \mu' L'}{a}, \quad \frac{\mu M c_2 - \mu' M'}{a}, \quad \frac{\mu N c_3 - \mu' N'}{a}\right)$$

respectively. Thus both projective centres are determined from the new ray origins by a displacement of magnitude μ'/a parallel to the refracted ray in the direction of ρ' increasing, and a displacement of magnitude μ/a parallel to the incident ray in the direction of ρ decreasing. The geometrical equivalence of all the transformed equations is now obvious.

When the system is telescopic the terms independent of x, y, z, x', y', z' may be removed by changes of origin, but the values of the b's and c's cannot be altered. In general, the traditional Smith-Helmholtz law will not apply to telescopic systems unless the projective centres are coplanar or collinear.

Modified equations

We can readily derive from the preceding equations alternative formulae for the computation of conjugate points and of the chief constants of the system in great variety. It will suffice to take a simple example. From the equation

$$\mu' x' L' - \mu x L + x x' K = 0,$$

if x_0 and x_0' are any pair of values of x and x' which satisfy the equation, we have

$$(x' - x_0')(\mu' L' + x_0 K) - (x - x_0)(\mu L - x_0' K) + (x - x_0)(x' - x_0') K = 0.$$

Particularly simple expressions for the power and other coefficients of a combination are obtained when the x-differences are unchanged on proceeding from one surface to the next. Examples of importance occur when the point of refraction or the centre of curvature for one refraction lies on the aplanatic surface of the next refracting surface.

We may apply the initial equation again to the modified form to obtain

$$\frac{\mu' L'}{x' - x_0'} \cdot \frac{x_0'}{x_0} - \frac{\mu L}{x - x_0} \cdot \frac{x_0}{x_0'} - K = 0.$$

By equations (4) this is evidently only another way of writing the corresponding modification of (3), viz.

$$\frac{1}{\mu' (\rho' - \rho_0')} \cdot \frac{\rho_0'}{\rho_0} - \frac{1}{\mu (\rho - \rho_0)} \cdot \frac{\rho_0}{\rho_0'} - \frac{K}{\mu \mu'} = 0.$$

The value of these equations lies in the fact that we may choose the ratios x_0'/x_0 and ρ_0'/ρ_0 arbitrarily. All the ordinary forms are obtained by making these ratios a function of the relative refractive index. For example, in the paraxial region, where both x and ρ are measured along the axis, we derive the x-equation (5) with $L = L' = 1$ by putting $\rho_0 = \rho_0' = r$ in the equation just given.

The primary equations

We have taken the single surface projection centres on the normal in this discussion because we are dealing with the Smith-Helmholtz equation, which gives the magnification ratio corresponding to secondary conjugate points. It would be a mistake, however, to suppose that projections through other points lead to relations of a different type. The equations are readily modified to meet other cases. We may, in fact, derive the equations for these other cases by starting from the secondary form of equation. As an illustration we may take the primary relation for a spherical surface. We only have to note that all the projections to be determined are fully known when three pairs of conjugate points are known. For the primary relation the point of refraction is a self-conjugate point, another pair of points projective from the centre of curvature are the aplanatic points, and the feet of the perpendiculars from the centre to the rays are also conjugate. By adding equations (5) we obtain

$$\mu' p' - \mu p + dd' K = 0,$$

where d and d' are the distances of (x, y, z) and (x', y', z') from the origin, and p and p' are the projections of d and d' on the corresponding parts of the ray. At the aplanatic points $p = d \cos \phi'$, $p' = d' \cos \phi$. It at once follows, since $p = 0$ and $p' = 0$ are to be conjugate, that the equation referred to the feet of the central perpendiculars as origins is

$$\mu' p' - \mu p + pp' H = 0,$$

where

$$H = K/\cos \phi \cos \phi'.$$

This is the primary equation corresponding most closely to the secondary equation (3).

If we take Young's primary projective point as origin, the resolved equations

$$\mu' x' L' \cos^2\phi' - \mu x L \cos^2\phi + x x' K = 0,$$
$$\mu' y' M' \cos^2\phi' - \mu y M \cos^2\phi + y y' K = 0,$$
$$\mu' z' N' \cos^2\phi' - \mu z N \cos^2\phi + z z' K = 0,$$

superficially bear a closer resemblance to the usual equation

$$\frac{\mu' \cos^2\phi'}{q'} - \frac{\mu \cos^2\phi}{q} - K = 0,$$

referred to the point of refraction as origin, than to the central form. To obtain matrices with unit determinants for compounding, these equations will be re-written in the standard form

$$\mu' x' L' \frac{\cos\phi'}{\cos\phi} - \mu x L \frac{\cos\phi}{\cos\phi'} + x x' H = 0,$$
$$\mu' y' M' \frac{\cos\phi'}{\cos\phi} - \mu y M \frac{\cos\phi}{\cos\phi'} + y y' H = 0,$$
$$\mu' z' N' \frac{\cos\phi'}{\cos\phi} - \mu z N \frac{\cos\phi}{\cos\phi'} + z z' H = 0,$$
$$q q' \left[\frac{\mu'}{q'} \frac{\cos\phi'}{\cos\phi} - \frac{\mu}{q} \frac{\cos\phi}{\cos\phi'} - H \right] = 0.$$

By proceeding as in the case of the secondary equation we may show that the coefficients of pp', qq', xx', yy', zz' will always be identical, as in the case of the single surface.

The resolved equations are far less useful than the corresponding secondary relations for the sole reason that the position of the primary projective centre cannot as a rule be determined by inspection. If the origin is moved from the projective centre to a known point, such as the centre of curvature, the direction cosines L and L' are not simply associated with the corresponding x and x', and the equations can hardly be regarded as members of the Smith-Helmholtz group.

The general transformation

It may be observed that all the variations of any equation for finding conjugate points may be brought under a single rule. The standard expression is always of the form

$$(1\ x)\begin{pmatrix} b & d \\ a & c \end{pmatrix}\begin{pmatrix} x' \\ 1 \end{pmatrix} = 0, \qquad \begin{vmatrix} b & d \\ a & c \end{vmatrix} = 1.$$

If this is the equivalent of

$$(1\ \xi)\begin{pmatrix} \beta & \delta \\ \alpha & \gamma \end{pmatrix}\begin{pmatrix} \xi' \\ 1 \end{pmatrix} = 0,$$

where $\quad \xi = ux + v, \quad \xi' = u'x' + v',$

we have $\quad \alpha\,(uu')^{\frac{1}{2}} = a, \quad \beta\gamma - \alpha\delta = 1,$

$$\frac{\beta}{\alpha} = u\frac{b}{a} - v, \quad \frac{\gamma}{\alpha} = u'\frac{c}{a} - v',$$

so that $\quad \frac{\beta}{\alpha} + \xi = u\left(\frac{b}{a} + x\right), \quad \frac{\gamma}{\alpha} + \xi' = u'\left(\frac{c}{a} + x'\right).$

A CHROMATIC METHOD OF MAKING OPTICAL SETTINGS

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ABSTRACT. The paper describes a visual modification of a previously described photo-electric method of measuring small angles. Great simplification is introduced by the substitution of visual for photo-electric equipment at the sacrifice of sensitivity. Angular displacements are measured by means of a hollow prism previously described. The settings are made by causing an image of a fine slit to fall on the sharp edge of a steel prism. The two components of the divided beam, after being coloured respectively red and green, are recombined, and a visual method is employed to observe the resulting colour. The relative position of the prism edge and image determines the proportion of red and green and is very sharply defined by the position of best resultant yellow. The settings have average errors of 0·2 second and are less fatiguing than cross-wire settings.

In a recent paper* the author describes a photo-electric equipment by means of which small angles may be measured with great accuracy. The present paper describes briefly an alternative visual method which does not indeed reach the same limits, but which is as accurate as and less fatiguing than that which employs cross-wire settings, and requires much simpler equipment than the photo-electric method. Unlike the photo-electric method, in which a scale of measurement is possible over a small range, the present method is essentially one of null indication only. This is not, however, a relative demerit, as the photo-electric method is intended only for null indication. The optical system is shown in the figure. The details are in the main identical with those described in the previous paper. The same micrometer is employed.

As before, an image of the slit S is formed on the edge of the stainless steel prism p by the lens L and mirror M. Between the prism p and the prisms P_1 and P_2 are respectively inserted red and green filters r and g. The coloured components, collimated by the lenses L_1, L_2, are partially reflected by the plane unsilvered glass mirrors m_1, m_2. The reflected beams are superimposed and reach the 45° prism P_4 which reflects them to the neighbourhood of the controls. The plates m_1, m_2 replace the totally reflecting prisms P_4, P_5 used in the previous method to direct the beams on to the photo-electric cell. The reason for employing plates in the present method is that exact superposition of beams is necessary, and is of course impossible to

* D. S. Perfect, "A method of measuring small angles." Phys. and Opt. Soc. Joint Discussion on Photo-electric Cells, 1930.

attain with the use of prisms. Further, the loss of energy which is inevitable when using plates and would seriously affect the sensitivity of the photo-electric method causes no inconvenience, as the visual field is quite sufficiently bright for its colour to be discriminated.

The light from the prism P_4 is directed on to a lens L_5, which focusses it on to a pinhole aperture E, through which an observer sees the whole aperture of L_5 approximately uniformly bright. The colour of the visual field, however, depends

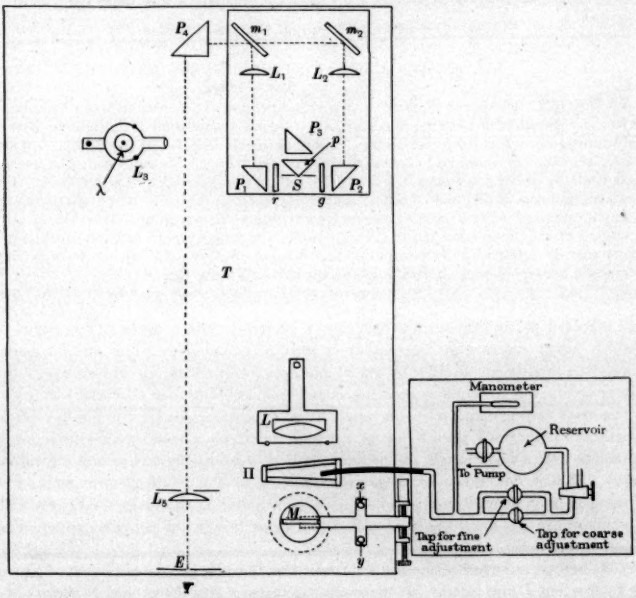

Fig. 1.

on the relative intensity of the red and green components present in the beam reaching L_5, that is to say, it depends on the relative position of the image of S and the edge of the prism p. The relative position for which the colour of the field appears most to match yellow is very sharply defined, and this appearance of the colour is therefore chosen as the criterion for making settings. When the image is well to one side of its normal or standard position the observer knows by the nature of the colour of the field in which direction to move the controls to bring the image back again.

The geometrical properties of various component parts are as follows: The steel table T measures 6 ft. × 3 ft. The slit S is 0·16 mm. in width. The lens L has a focal length of 125 cm. The focal length of each of the lenses L_1, L_2 is 19·8 cm. The focal length of L_3 is 14·3 cm., and the stop limiting its aperture is 1 cm. in diameter.

By means of this apparatus settings can be made with very little personal fatigue, with probable error in angle of the order of 0·2 second. The condition most difficult to fulfil is the attainment of uniformity of the visual field. Since the edge of the dividing prism is in fact performing a similar function to the knife-edge in a knife-edge test, correct focussing and perfection of optical parts is essential. In the best conditions hitherto secured in practice the uniformity of the field is marred by slight striations which spoil the colour blending. If a sensibly uniform field could be produced there is little doubt that the precision of the method would be much further increased.

ON THE DIFFRACTION OF LIGHT BY A SLIT

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ABSTRACT. Next to the problem of the straight edge the simplest and most important problem in diffraction is that of the slit. The former, known as the wedge problem, has been solved. The latter still defies solution.

Admittedly the elementary solution of the problem of the slit is sufficient as an explanation of the phenomena commonly observed, but as an exposition of wave theory it leaves much to be desired. An attempt is here made to discuss analytically the theory of the slit as defined below.

A slit is a narrow infinitely long opening in an infinitely thin plane screen. The term narrow is relative, meaning that the width is small compared with the distance at which the phenomena are observed, but large in comparison with the wave-length of the incident light.

The resulting analysis is applicable to points far inside the geometrical shadow, with which the elementary theory cannot cope.

Introduction

The problem of the diffraction of waves by a straight edge, known as the problem of the wedge, is one of the most important problems in mathematical physics which has been solved exactly, and in recent years it has received a very great amount of attention. The reason lies partly in its test of the accuracy of Fresnel's approximate formulae, and partly in its analytical interest.

The problem of the slit is of at least as great physical and analytical importance. The theory of the transmission of plane waves through apertures in plane screens has been treated by Rayleigh upon the assumption that the linear dimensions of the aperture are small compared with the wave-length of the incident waves. The theory of the slit is also considered when the width of the slit is small in comparison with the wave-length (Rayleigh, "On the Passage of Waves through Apertures in Plane Screens, and Allied Problems," *Coll. Papers*, vol. IV, pp. 291–4). The classical theory of diffraction by a slit when the width is great in comparison with the wave-length is satisfactory only in so far as it serves to explain the most commonly observed phenomena of diffraction.

In the present paper I have solved the problem for small wave-lengths in a manner analogous to that used by Rayleigh in the consideration of large wave-lengths. I have also considered briefly the classical theory, which may be called a first approximation since it assumes that the surface of resolution is unaffected by diffraction due to the edges of the slit.

In a summary I have set out for comparison the three most important results of theory for large wave-lengths, and the corresponding three for small wave-lengths.

In a recent paper (*Phil. Trans. Roy. Soc.* Series A, **229**) I have considered some general methods of dealing with a class of diffraction problems. I am not yet quite satisfied that those methods are competent to give the complete solution of the problem of the diffraction of waves by a slit, a problem hitherto unsolved. The theory of the slit is, accordingly, considered in the present paper from a different standpoint.

In the problem of the straight edge I have shown, in the paper referred to, that energy is propagated into the region far inside the shadow as though from sources situated upon the straight edge.

In the present paper it is shown generally that, in the case of the slit, the effects far inside the shadow are mainly due to the edges of the slit. It follows that in those regions the formulae of diffraction should be modified by the nature of the disturbance close to the edges of the slit. Formulae which are valid throughout the whole of space at great distances from the slit are evolved. These include Fraunhofer's approximate diffraction formulae, and, so far as I know, they are the first to indicate definitely to what extent the main assumptions, which are made in the classical treatment of diffraction by a slit, are valid.

In practical optics the most important case to come under review is that in which the width of the slit is, at least, moderately great in comparison with the wave-length of the incident light. The theory of the present paper is applicable without serious error to all cases in which the width of the slit is greater than about two wave-lengths.

In order to simplify the analysis still further attention is confined to those phenomena investigated by Fraunhofer, in which the secondary beams are brought to a principal focus. This simplification is equivalent to applying the theory to points at a distance from the slit which is great in comparison with the width of the slit.

Finally, in all cases, the incident waves are assumed to be plane waves falling normally upon the slit.

Preliminary Theory

For the preliminary analysis no attention need be paid to the boundary conditions. If the normal to the incident wave-front, supposed plane, be perpendicular to the straight edges of the slit, then the surface of resolution or, in other words, the source of the secondary waves of Huygen's is a system of parallel line sources.

Kirchhoff's theorem, for a single source, is expressed in the following form.

A simple harmonic source S, denoted by $Ae^{i\kappa ct}$, is assumed. In this expression A is the amplitude, c the velocity of light, and $2\pi/\kappa$ $(= r)$ the wave-length of the emitted simple harmonic wave. The real part of the result corresponds to a real source $A \cos \kappa ct$.

A closed surface Σ is also postulated, which does not enclose the source. It is required to find the effect at a point P within Σ.

Let r_1 be the distance from S to an element $d\Sigma$ of the closed surface, and r that from this element to P. Let $F = Ae^{i\kappa(ct-r-r_1)}$.

The effect at P, expressed by U_0, can be represented by the surface integral

$$4\pi U_0 = \iint \left\{ \frac{1}{r} \frac{\partial}{\partial r_1} \left(\frac{F}{r_1} \right) \cos(nr_1) - \frac{1}{r_1} \frac{\partial}{\partial r} \left(\frac{F}{r} \right) \cos(nr) \right\} d\Sigma \qquad \ldots\ldots(1),$$

where n is the normal drawn into the space enclosed by the surface. The proof of equation (1) will be found in many text-books.

Consider now an infinitely long element AB of a surface of resolution bounded by the edges of the slit (Fig. 1). Let its width be dx_1 and let OX be perpendicular to AB and tangential to the surface of resolution at O.

The axis of z is taken to coincide with AB. Let the source S lie in the plane of xy, and let it be required to find the effect at a point P which also lies in the plane of xy. Let Q be a point on AB distant r_1 and r from S and P respectively.

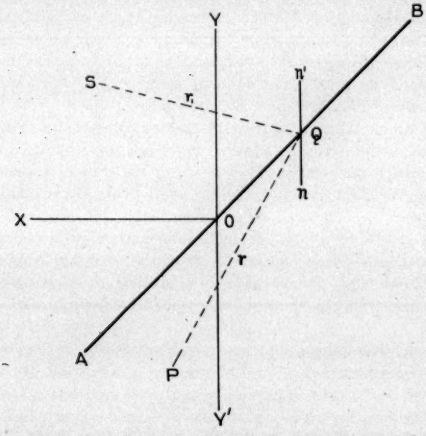

Fig. 1.

Let the normal at O to the surface of resolution, drawn inwards along OY', be denoted by m, and let $OS = \rho_1$ and $OP = \rho$. We have

$$\cos(nr_1) = \cos OSQ \cos(m\rho_1),$$

and

$$\cos(nr) = \cos OPQ \cos(m\rho).$$

If $OQ = z$ we obtain, for that part of U_0 due to the linear element AB, from (1)

$$4\pi dU_0 = 2 \int_0^\infty \left\{ \frac{1}{r} \frac{\partial}{\partial r_1} \left(\frac{F}{r_1} \right) \cos OSQ \cos(m\rho_1) - \frac{1}{r_1} \frac{\partial}{\partial r} \left(\frac{F}{r} \right) \cos OPQ \cos(m\rho) \right\} dz\, dx$$

The differentiations involved are along the normal, so that z remains constant. Hence

$$4\pi dU_0 = 2 \int_0^\infty \left\{ \frac{1}{r} \frac{\partial}{\partial \rho_1} \left(\frac{F}{r_1} \right) \cos(m\rho_1) - \frac{1}{r_1} \frac{\partial}{\partial \rho} \left(\frac{F}{r} \right) \cos(m\rho) \right\} dz\, dx.$$

Now, since the incident wave is assumed to be plane, $Ae^{-i\kappa r_1}/r_1$ may be replaced by $Be^{i\kappa\sigma}$, where σ is the perpendicular distance from the linear element AB to the incident wave-front. Also $\partial/\partial\rho_1$ may be replaced by $-\partial/\partial\sigma$. Hence, for the present omitting the time factor $e^{i\kappa ct}$ in F,

$$4\pi dU_0 \to -2B\int_0^\infty e^{i\kappa\sigma}\left\{i\kappa\frac{1}{r}e^{-i\kappa r}\cos(m\sigma)+\frac{\partial}{\partial\rho}\left(\frac{e^{-i\kappa r}}{r}\right)\cos(m\rho)\right\}dz\,dx.$$

Now
$$\int_0^\infty \frac{e^{-i\kappa r}}{r}dz=\int_0^\infty e^{-i\kappa\rho\cosh u}du=K_0(i\kappa\rho).$$

Also
$$-\partial K_0(i\kappa\rho)/\partial\rho=i\kappa K_1(i\kappa\rho).$$

The theory of the functions K_0 and K_1 will be found in Whittaker and Watson's *Modern Analysis*, chap. XVII. They are important in a number of physical applications. For $K_0(i\kappa\rho)$ Lamb uses $\frac{1}{2}\pi D_0(\kappa\rho)$ [*Hydrodynamics*, 4th ed., chap. VIII], and it is identical with Rayleigh's function $-\psi_0$ [*Theory of Sound*, **2**, 305].

The above result may now be replaced by

$$4\pi dU_0 \to 2Bi\kappa e^{i\kappa\sigma}\{K_1(i\kappa\rho)\cos(m\rho)-K_0(i\kappa\rho)\cos(m\sigma)\}dx.$$

The functions $K_0(i\kappa\rho)$ and $K_1(i\kappa\rho)$ each have the asymptotic value

$$\left(\frac{\pi}{2i\kappa\rho}\right)^{\frac{1}{2}}e^{-i\kappa\rho}.$$

Hence, for large values of $\kappa\rho$,

$$4\pi dU_0 \to 2Bi\kappa\left(\frac{\pi}{2i\kappa\rho}\right)^{\frac{1}{2}}e^{i\kappa(\sigma-\rho)}\{\cos(m\rho)-\cos(m\sigma)\}dx \qquad \text{.....(2).}$$

The complete analysis of a line element in relation to diffraction is not developed with sufficient rigidity in text-books of optics.

As the distance from the slit, at which the effect is desired, increases, formula (2) tends to become exact. An interesting verification of this fact will be obtained later.

As is well known, the factor $i^{-\frac{1}{2}}$ in (2), or its equivalent $e^{-i\frac{1}{4}\pi}$, represents a phase difference of one-eighth of a wave-length as compared with the corresponding expression for an element $d\Sigma$.

The application of (2) to a slit will now be considered. A surface of resolution joining the edges of the slit is, obviously, finite in width and, if $\kappa\rho$ becomes sufficiently great, the factor $(\kappa\rho)^{-\frac{1}{2}}$ may be taken outside the range of integration. This is the process employed in treating Fraunhofer's diffraction phenomena and (2), under these circumstances, takes the form

$$dU_0=Ce^{i\kappa(\sigma-\rho)}\{\cos(m\rho)-\cos(m\sigma)\}dx,$$

where C is a constant so far as integration over the surface of resolution is concerned. Diffraction is due to the path differences from the secondary sources so that, if Δ be the difference of path from any given source and that from one edge O (Fig. 2) of the slit, we may write

$$\sigma-\rho=\sigma_0-\rho_0+\Delta.$$

It follows finally that

$$U_0=i\kappa C_0\int e^{i\kappa\Delta}\{\cos(m\rho)-\cos(m\sigma)\}dx \qquad \text{.....(3),}$$

in which the integration is effected over a line joining the edges of the slit and

$$C_0 = \frac{B}{2\pi}\left(\frac{\pi}{2\iota\kappa\rho_0}\right)^{\frac{1}{2}} e^{-\iota\kappa\rho_0},$$

ρ_0 being the distance from O to the point under consideration.

It must be noted that dx is a linear element of the surface of resolution, and that it does not necessarily lie in a plane containing the edges of the slit.

The Theory of the Slit: First Approximation

This approximation involves the exact use of formula (3). It is an approximation because it takes no account of the effect of diffraction upon the surface of resolution.

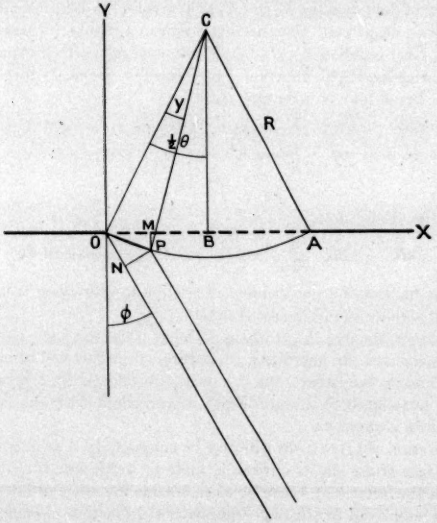

Fig. 2.

Instead of applying (3) to the plane surface of the slit, which is always done, another process is here adopted which, for a finite surface of resolution, embodies, so far as I know, the only verification by direct integration of Kirchhoff's important theorem. It is assumed that the surface of resolution in the case of the slit is any one of a family of cylinders with circular sections, whose centres lie upon the perpendicular to the line joining the edges of the slit and midway between them.

Let OA (Fig. 2) be the section of the slit, and let the arc OPA of the circle, whose centre C is situated on the perpendicular through the centre B of the slit, be taken as the surface of resolution.

Waves are incident normally upon OA, and we consider Fraunhofer's diffraction phenomena, in which the bands are observed upon the focal plane of an objective.

Taking P as some point upon the surface of resolution, let

$$OCP = \gamma \text{ and } OCB = \tfrac{1}{2}\theta.$$

Draw PM perpendicular to OA and PN perpendicular to a line from O drawn in the direction ϕ, the angle of diffraction. The phase difference to be evaluated is due to the path difference

$$\begin{aligned}\Delta &= ON - MP \\ &= OP\,(\cos PON - \sin POM).\end{aligned}$$

Now, if the radius of the circle of resolution be R,

$$OP = 2R \sin \tfrac{1}{2}\gamma.$$

It follows that, denoting the phase difference by D, the latter is given by

$$\begin{aligned}D &= \kappa\Delta \\ &= 2\kappa R \sin \tfrac{1}{2}\gamma \{\sin(\phi + \tfrac{1}{2}\theta - \tfrac{1}{2}\gamma) - \sin(\tfrac{1}{2}\theta - \tfrac{1}{2}\gamma)\} \\ &= 2\kappa R \sin \tfrac{1}{2}\phi \{\sin(\tfrac{1}{2}\phi + \tfrac{1}{2}\theta) + \sin(\gamma - \tfrac{1}{2}\phi - \tfrac{1}{2}\theta)\}.\end{aligned}$$

Again, $$(m\rho) = \phi + \tfrac{1}{2}\theta - \gamma$$

and $$(m\sigma) = \pi - \tfrac{1}{2}\theta + \gamma.$$

Hence $$\cos(m\rho) - \cos(m\sigma) = 2\cos\tfrac{1}{2}\phi \cos(\gamma - \tfrac{1}{2}\phi - \tfrac{1}{2}\theta).$$

The integral required is, accordingly, from (3),

$$U_0 = 2C_0 R \cos\tfrac{1}{2}\phi \int_{\gamma=0}^{\gamma=\theta} \exp[2i\kappa R \sin\tfrac{1}{2}\phi \{\sin(\tfrac{1}{2}\phi + \tfrac{1}{2}\theta) + \sin\epsilon\}] \cos\epsilon . d\gamma,$$

where $$\epsilon = \gamma - \tfrac{1}{2}\phi - \tfrac{1}{2}\theta.$$

Now $$\begin{aligned}\cos\epsilon . d\gamma &= \cos\epsilon . d\epsilon \\ &= d(\sin\epsilon).\end{aligned}$$

Hence $$U_0 = 2C_0 \cot\tfrac{1}{2}\phi \{e^{i\kappa \sin\phi . 2R \sin\frac{1}{2}\theta} - 1\} \qquad \ldots\ldots(4).$$

This expression is independent of θ, since

$$2R \sin\tfrac{1}{2}\theta = 1,$$

the width of the slit being assumed to be unity. If I_0 be the relative intensity, referred to unit value when $\phi = 0$, then it follows in the usual way that

$$\begin{aligned}I_0 &= \frac{1}{4\kappa^2} \cot^2\tfrac{1}{2}\phi \{\sin^2(\kappa \sin\phi) + (\cos\kappa \sin\phi - 1)^2\} \\ &= \cos^2\tfrac{1}{2}\phi . \frac{\sin^2(\tfrac{1}{2}\kappa \sin\phi)}{\kappa^2 \sin^2\tfrac{1}{2}\phi} \qquad \ldots\ldots(5).\end{aligned}$$

This result leads to the classical expression for the determination of the maxima in Fraunhofer's diffraction phenomena when ϕ is small, viz.

$$\tan(\tfrac{1}{2}\kappa\phi) = \tfrac{1}{2}\kappa\phi.$$

The foregoing analysis verifies the fact that the result at a point remains the same, whatever the surface of resolution, so long as all the surfaces are bounded by the same closed curve, for in the case of Fraunhofer's phenomena the form (3) of Kirchhoff's theorem is exact.

General Theory of the Slit

A straightforward application of Huygen's principle is sufficient to explain the more salient features of the phenomena observed. The accuracy of the theory is, however, based upon little more than its agreement with experiment, so that a more rigorous treatment is essential, in which the optical properties of the boundaries of the slit must be taken into account.

The boundaries of the slit are assumed to be infinitely thin planes and plane electro-magnetic waves are incident normally.

In Fig. 2 a section of the slit is shown, O and A being the edges. The axis of y is drawn normally from O, and the axis of z coincides with the edge of the slit through O. The plane $y = 0$ coincides with the two infinitely thin boundaries and the surface of resolution.

Let the width OA of the slit be taken as unity as before, and let $\kappa = 2\pi/\lambda$, where λ is the wave-length of the incident wave. Fraunhofer's diffraction phenomena, as will be shown, depend upon a function

$$u = A\int_0^1 f(x)\, e^{i\kappa x \sin\phi}\, dx \qquad \text{.....(6),}$$

where A is a function of ρ and ϕ, the coordinates of a point at great distances from the slit, and the integral is taken over the section OA.

Before proceeding to the determination of the function $f(x)$, some general results are required.

The right-hand side of (6) may be expanded in two forms. In the first place

$$\int f(x)\, e^{i\kappa x \sin\phi}\, dx = e^{i\kappa x \sin\phi} f_1(x)\left\{1 - i\kappa \sin\phi \frac{f_2(x)}{f_1(x)} + (i\kappa \sin\phi)^2 \frac{f_3(x)}{f_1(x)} - \ldots\right\} \qquad \text{.....(7),}$$

where $df_1/dx = f, \quad df_2/dx = f_1$, etc.

If κ be small, $u = A\left|f_1(x)\right|_0^1$,

approximately.

In the case of sound waves $f_1(x)$ depends upon the flow through the slit per unit length. Its form has been determined by Rayleigh in his paper upon the transmission of plane waves through small apertures in plane screens.

An integral may also be obtained in the form

$$\int f(x)\, e^{i\kappa x \sin\phi}\, dx$$

$$= \frac{1}{i\kappa \sin\phi} f(x)\, e^{i\kappa x \sin\phi}\left\{1 - \frac{1}{i\kappa \sin\phi}\frac{f'(x)}{f(x)} + \frac{1}{(i\kappa \sin\phi)^2}\frac{f''(x)}{f(x)} - \ldots\right\} \text{...(8),}$$

where $f'(x) = df/dx, \quad f''(x) = d^2f/dx^2$, etc.

If $\kappa \sin\phi$ be large, $u = \dfrac{A}{i\kappa \sin\phi}\left|f(x)\, e^{i\kappa x \sin\phi}\right|_0^1$ .....(9),

approximately.

If κ, then, be large and ϕ not too small, that is in the space well within the shadow, the expression (9) is the true approximation to the exact result. In this

case it is clear that the effects well within the shadow are due to the edges, for they depend upon the edge values of $f(x)$.

If, alternatively, $f(x)$ be a uniform constant so that $f'(x)$ and the subsequent derivatives vanish, the expression (9) is the requisite result at all points. It is proposed to find to what extent the usual assumption of uniformity is valid.

For this purpose let us consider that $f(x)$ consists of a uniform part and of a non-uniform part, and that the latter part sensibly vanishes at a distance b from each edge. Denoting this part by $F(x)$ near one edge and by $G(x)$ near the other, we have

$$u = A\left[\int_0^1 e^{i\kappa x \sin\phi}\,dx + \int_0^b F(x)\, e^{i\kappa x \sin\phi}\,dx + \int_{1-b}^1 G(x)\, e^{i\kappa x \sin\phi}\,dx\right].$$

The third term within the brackets may be written

$$\int_0^b G(1-x)\, e^{i\kappa(1-x)\sin\phi}\,dx,$$

in which, from symmetry in perpendicular incidence,

$$G(1-x) = F(x).$$

When κ is large the limit b may be replaced by ∞ ; we then have, expressed in the most suitable form for our purpose,

$$u/A = \int_0^1 e^{i\kappa x \sin\phi}\,dx + \int_0^\infty F(x)\left(e^{i\kappa x \sin\phi} + e^{-i\kappa x \sin\phi}\right)dx + \left(e^{i\kappa \sin\phi} - 1\right)\int_0^\infty F(x)\, e^{i\kappa x \sin\phi}\,dx \quad \ldots\ldots(10).$$

With regard to the approximations, the first neglects only the effect of diffraction upon the surface of resolution.

The second approximation includes a first order effect of diffraction upon the surface of resolution. This may be seen as follows:

A plane wave falling normally upon the slit affects the surface of resolution by diffraction from the edge O (Fig. 2). The wave falling upon the edge A differs slightly from a plane wave owing to the small residual disturbance due to diffraction from the edge O. We neglect the diffraction produced by the edge A upon the surface of resolution as a result of this small residual disturbance.

So far as the incident waves are concerned it is proposed to consider two cases.

Case A. The polarisation in the incident wave-front is parallel to the edges of the slit.

Case B. The polarisation in the incident wave-front is perpendicular to the edges of the slit.

Case A. The incident wave is denoted by $e^{i\kappa(y+ct)}$, and in this case the solution must vanish over both sides of the screens which form the boundaries of the slit.

At any point on that side of the screens away from the incident wave the function has the form

$$u = \frac{1}{2\pi}\iint u\,\frac{\partial}{\partial n}\left(\frac{e^{-i\kappa r}}{r}\right)dS \qquad \ldots\ldots(11).$$

This formula is rigidly applicable to the present case, since u vanishes at all points of the boundaries. The integration is accordingly confined to the plane surface of the slit.

For the proof of (11) reference must be made for example to Lamb's *Hydrodynamics*, 4th ed., p. 493.

In the equation (11) r is the distance of the point from the element dS of the slit and n is the normal drawn into the space away from the incident wave.

Equation (11) may be written

$$u = \frac{1}{2\pi}\iint u \frac{\partial \rho}{\partial n}\cdot\frac{\partial}{\partial \rho}\left(\frac{e^{-ikr}}{r}\right) dS.$$

Hence for the slit (see preliminary theory)

$$u = -\frac{ik}{2\pi}\int_0^1 u \frac{\partial \rho}{\partial n} K_1(ik\rho)\, dx.$$

When ρ is large this becomes

$$u = -\frac{ik}{2\pi}\left(\frac{\pi}{2ik\rho_0}\right)^{\frac{1}{2}} e^{-ik\rho_0 \cos\phi}\int_0^1 f(x)\, e^{ikx \sin\phi}\, dx \qquad \ldots\ldots(12).$$

Case B. Denoting the incident wave as before by $e^{ik(y+ct)}$, the condition to be satisfied is that $\partial u/\partial n$ must vanish over both sides of the screens which bound the slit. At any point on that side of the screens away from the incident wave the function u has the form

$$u = -\frac{1}{2\pi}\iint \frac{e^{-ikr}}{r}\cdot\frac{\partial u}{\partial n}\, dS \qquad \ldots\ldots(13).$$

This formula is rigidly applicable to the present case, since $\partial u/\partial n$ vanishes at all points of the boundaries. The integration is accordingly confined to the plane surface of the slit.

For the proof of (13) reference must be made again to Lamb's *Hydrodynamics*, 4th ed., p. 493. For the slit (see preliminary theory)

$$u = -\frac{1}{2\pi}\int_0^1 \frac{\partial u}{\partial n} K_0(ik\rho)\, dx.$$

When ρ is large this becomes

$$u = -\frac{1}{2\pi}\left(\frac{\pi}{2ik\rho}\right)^{\frac{1}{2}} e^{-ik\rho}\int_0^1 f(x)\, e^{ikx \sin\phi}\, dx \qquad \ldots\ldots(14).$$

The functions f of (12) and (14) are, of course, not the same.

We have now to consider the nature of these functions, and for this purpose recourse must be had to another problem.

The Straight Edge or Semi-infinite Plane

This problem is a special case of the problem of the perfectly reflecting wedge, the most general treatment of which is due to Bromwich (*Proc. London Math. Soc.* Ser. 2, **14**, Part 6) and Macdonald (*ibid.*). If there is a point source from which waves diverge, and if this source lies in the space contained between two planes which make an angle α with one another, the problem can be solved directly by the method of images if α is a submultiple of two right angles.

The summation of the effects due to the source and its images can be replaced by a contour integral in the domain of a complex variable.

If $\alpha = \pi/n$, any wedge can be represented by giving n all values from ∞ to $\frac{1}{2}$. The latter value corresponds to the semi-infinite plane.

By choosing a suitable contour a solution can be found which is valid for all

values of n, and thus analytically the problem of regular reflection merges into that of diffraction.

Since the problem is so important a method, which seems to be the simplest possible, is here given for arriving at the solution of the special problem required in this paper.

The incident wave is again represented by

$$e^{i\kappa(y+ct)}.$$

Omitting the time factor, for the present, we commence by considering solutions of the two-dimensional wave-equation of the form

$$F_0 = e^{i\kappa y} F_1.$$

In this equation F_0 is a wave-function so that it satisfies the equation

$$(\nabla^2 + \kappa^2) F_0 = 0,$$

where ∇^2 stands for $\partial^2/\partial x^2 + \partial^2/\partial y^2$.

It follows that F_1 satisfies the equation

$$\nabla^2 F_1 + 2i\kappa \frac{\partial F_1}{\partial y} = 0 \qquad \text{.....(15).}$$

A general transformation of the variables x and y is now useful. This transformation is the orthogonal one

$$y + ix = f(\phi + i\psi) \qquad \text{.....(16),}$$

and the form of f must be chosen to suit the particular problem in hand.

From (16) it follows that x and y are functions of ϕ and ψ, and it is not difficult to show that (15) transforms into

$$\frac{\partial^2 F_1}{\partial \phi^2} + \frac{\partial^2 F_1}{\partial \psi^2} + 2i\kappa \left\{ \frac{\partial F_1}{\partial \phi} \cdot \frac{\partial x}{\partial \psi} - \frac{\partial F_1}{\partial \psi} \cdot \frac{\partial x}{\partial \phi} \right\} = 0 \qquad \text{.....(17).}$$

Let O (Fig. 3) be the edge of the semi-infinite plane, and let r, θ_1 be the polar coordinates of a point P. We endeavour to find a function F_1 which satisfies (15) and which has a period 4π in θ_1, being therefore single-valued in a Riemann space of two sheets. For this purpose the transformation at once suggested is the parabolic one

$$y + ix = \tfrac{1}{2}(\phi + i\psi)^2,$$

so that (17) becomes

$$\frac{\partial^2 F_1}{\partial \phi^2} + \frac{\partial^2 F_1}{\partial \psi^2} + 2i\kappa\phi \frac{\partial F_1}{\partial \phi} - 2i\kappa\psi \frac{\partial F_1}{\partial \psi} = 0 \qquad \text{.....(18).}$$

From the parabolic transformation we have

$$r + y = \phi^2, \quad \text{and} \quad r - y = \psi^2,$$

so that

$$\phi = (2r)^{\frac{1}{2}} \cos \tfrac{1}{2}\theta_1.$$

Hence ϕ is of the period required in θ_1. The function F_1 is therefore a solution of

$$\frac{\partial^2 F_1}{\partial \phi^2} + 2i\kappa\phi \frac{\partial F_1}{\partial \phi} = 0,$$

and the solution is

$$F_1 = A \int_B^{\phi} e^{-i\kappa u^2} du.$$

If OA (Fig. 3) be the semi-infinite plane, then the solution

$$F_1 = \pi^{-\frac{1}{2}} e^{i\frac{1}{4}\pi} \int_{-\infty}^{\alpha} e^{-iv^2} dv,$$

where

$$\alpha = (2\kappa r)^{\frac{1}{2}} \cos \tfrac{1}{2}\theta_1,$$

tends to unity everywhere at great distances outside the geometrical shadow $\theta_1 = \pi$, and tends to vanish at great distances within the geometrical shadow.

If we write I for the wave-function appertaining to the incident wave,

$$I = \pi^{-\frac{1}{2}} e^{i\kappa r \cos(\theta - \theta_0) + i\frac{1}{4}\pi} \int_{-\infty}^{\alpha} e^{-iv^2} dv \qquad \text{.....(19),}$$

where $$\alpha = (2\kappa r)^{\frac{1}{2}} \cos \tfrac{1}{2}(\theta - \theta_0).$$

If H be the wave-function appertaining to the reflected wave,

$$H = \pi^{-\frac{1}{2}} e^{i\kappa r \cos(\theta + \theta_0) + i\frac{1}{4}\pi} \int_{-\infty}^{\beta} e^{-iv^2} dv \qquad \text{.....(20),}$$

where $$\beta = (2\kappa r)^{\frac{1}{2}} \cos \tfrac{1}{2}(\theta + \theta_0).$$

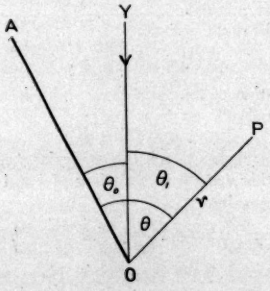

Fig. 3.

It may be noted that H tends to vanish at great distances outside the geometrical shadow $\theta + \theta_0 = \pi$ of the reflected wave. For perpendicular incidence $\theta_0 = \frac{1}{2}\pi$, and, when the width of the slit is moderately large compared with the wave-length of the incident wave, the function $F(x)$ of equation (10) can be obtained at once from the solution for the semi-infinite plane.

The Theory of the Slit. Second Approximations

Case A. When the polarisation in the incident wave-front is parallel to the edge of the semi-infinite plane the solution is

$$I - H.$$

Remembering that the x-coordinate of equation (10) is measured from O over the surface of the slit, we put $\theta_0 = \frac{1}{2}\pi$, $\theta = \pi$, and $r = x$ in (19) and (20), whence $I - H$ reduces to

$$2\pi^{-\frac{1}{2}} e^{i\frac{1}{4}\pi} \int_0^{(\kappa x)^{\frac{1}{2}}} e^{-iv^2} dv.$$

This expression is the effect of diffraction by the edge O over the surface of

resolution, and, as $\kappa x \to \infty$, its value $\to$ unity. Hence $F(x)$ of equation (10) becomes

$$F(x) = -2\left(\frac{i}{\pi}\right)^{\frac{1}{2}} \int_{(\kappa x)^{\frac{1}{2}}}^{\infty} e^{-iv^2}\,dv \qquad \text{.....(21).}$$

Again from equation (10) we have to consider an integral of the form

$$G = \int_0^{\infty} F(x)\, e^{i\kappa x\beta}\,dx$$
$$= -\int_0^{\infty} e^{i\kappa x\beta} \cdot 2\left(\frac{i}{\pi}\right)^{\frac{1}{2}} \int_{(\kappa x)^{\frac{1}{2}}}^{\infty} e^{-iv^2}\,dv\,dx,$$

where $$\beta = \pm \sin\phi.$$

Writing $u(\kappa x)^{\frac{1}{2}}$ for v, it follows that

$$G = -2\left(\frac{i\kappa}{\pi}\right)^{\frac{1}{2}} \int_1^{\infty} du \int_0^{\infty} e^{-i\kappa x(u^2-\beta)} \cdot x^{\frac{1}{2}}\,dx.$$

The second integral in this expression has the value

$$\frac{\Gamma\left(\frac{3}{2}\right)}{\{i\kappa\,(u^2-\beta)\}^{\frac{3}{2}}},$$

where the gamma function $$\Gamma\left(\frac{3}{2}\right) = \tfrac{1}{2}\pi^{\frac{1}{2}}.$$

Hence $$G = -\frac{1}{i\kappa}\int_1^{\infty} \frac{du}{(u^2-\beta)^{\frac{3}{2}}} = \frac{1}{i\kappa\beta}\{1-(1-\beta)^{-\frac{1}{2}}\} \qquad \text{.....(22)}$$

It is not difficult to show that

$$\int_0^{\infty} F(x)\,(e^{i\kappa x\sin\phi} + e^{-i\kappa x\sin\phi})\,dx = -\frac{1}{i\kappa\cos\phi}\{(1+\sin\phi)^{\frac{1}{2}} - (1-\sin\phi)^{\frac{1}{2}}\}$$
$$= -\frac{2}{i\kappa\cos\phi}\sin\tfrac{1}{2}\phi.$$

The complete expression for u/A (equation (10)) is now, by a simple reduction,

$$\frac{u}{A} = i\kappa\left[\frac{e^{i\kappa\sin\phi}-1}{\sin\phi\,(1+\sin\phi)^{\frac{1}{2}}} - \frac{2\sin\tfrac{1}{2}\phi}{\cos\phi}\right],$$

and therefore, from (12),

$$u = -\frac{1}{2\pi}\left(\frac{\pi}{2i\kappa\rho_0}\right)^{\frac{1}{2}} e^{-i\kappa\rho_0}\left[\frac{\cot\phi}{(1+\sin\phi)^{\frac{1}{2}}}\,(e^{i\kappa\sin\phi}-1) - 2\sin\tfrac{1}{2}\phi\right] \qquad \text{...(23).}$$

The term $2\sin\frac{1}{2}\phi$ within the square brackets may be omitted if κ is large and if ϕ, the angle of diffraction, be not too great. But it must be retained in other circumstances. If these conditions are satisfied and if ρ be assumed to be constant, we may write for the relative intensity

$$I_0 = \frac{\cos^2\phi}{1+\sin\phi} \cdot \frac{\sin^2(\frac{1}{2}\kappa\sin\phi}{(\frac{1}{2}\kappa\sin\phi)^2} \qquad \text{.....(24),}$$

the constant factor having been adjusted so as to make I_0 unity when $\phi = 0$.

Case B. When the polarisation in the incident wave-front is perpendicular to the edge of the semi-infinite plane the solution is

$$I + H,$$

and from (13) we require the value of

$$\frac{\partial}{\partial n}(I+H)$$

over the surface of the slit.

When $\theta = \pi$, as before, and x is measured along the surface of resolution from O (Fig. 2), it is not difficult to show that

$$\frac{\partial}{\partial n}(I + H) = 2i\kappa \left(\frac{i}{\pi}\right)^{\frac{1}{2}} \int_0^{(\kappa x)^{\frac{1}{2}}} e^{-iv^2} dv + \left(\frac{i}{\pi}\right)^{\frac{1}{2}} \left(\frac{\kappa}{x}\right)^{\frac{1}{2}} e^{-i\kappa x}.$$

In this case $F(x)$ of equation (10) is given by

$$F(x) = i\kappa \left[-2\left(\frac{i}{\pi}\right)^{\frac{1}{2}} \int_{(\kappa x)^{\frac{1}{2}}}^{\infty} e^{-iv^2} dv + \frac{1}{i\kappa}\left(\frac{i}{\pi}\right)^{\frac{1}{2}} \left(\frac{\kappa}{x}\right)^{\frac{1}{2}} e^{-i\kappa x} \right] \quad \ldots\ldots(25).$$

The evaluation of the integral

$$\int_0^{\infty} F(x)\, e^{i\kappa x\beta}\, dx$$

presents no great difficulty.

The first term on the right-hand side of (25) is the same as that already dealt with. The second term involves an integral of a well-known type occurring in the theory of the gamma function.

We obtain

$$\int_0^{\infty} F(x)\, e^{i\kappa x \sin\phi}\, dx = \frac{1}{\sin\phi}\left\{1 - \frac{1}{(1-\sin\phi)^{\frac{1}{2}}}\right\} + \frac{1}{(1-\sin\phi)^{\frac{1}{2}}},$$

$$\int_0^{\infty} F(x)\, e^{-i\kappa x \sin\phi}\, dx = -\frac{1}{\sin\phi}\left\{1 - \frac{1}{(1+\sin\phi)^{\frac{1}{2}}}\right\} + \frac{1}{(1+\sin\phi)^{\frac{1}{2}}}.$$

By a little reduction

$$\int_0^{\infty} F(x)\left(e^{i\kappa x \sin\phi} + e^{-i\kappa x \sin\phi}\right) dx = \frac{1}{\cos\frac{1}{2}\phi}.$$

The complete expression for u/A (equation (10)) is now

$$\frac{u}{A} = \frac{e^{i\kappa\sin\phi} - 1}{\sin\phi}(1+\sin\phi)^{\frac{1}{2}} + \frac{1}{\cos\frac{1}{2}\phi},$$

and therefore, from (14),

$$u = -\frac{1}{2\pi}\left(\frac{\pi}{2i\kappa\rho_0}\right)^{\frac{1}{2}} e^{-i\kappa\rho_0}\left[\frac{(1+\sin\phi)^{\frac{1}{2}}}{\sin\phi}\left(e^{i\kappa\sin\phi} - 1\right) + \frac{1}{\cos\frac{1}{2}\phi}\right] \quad \ldots(26).$$

The term $1/\cos\frac{1}{2}\phi$ within the square brackets may be omitted if κ is large and if ϕ, the angle of diffraction, is not too great. But it must be retained in other circumstances. If these conditions are satisfied and if ρ be assumed to be constant, we may write for the relative intensity

$$I_0 = (1+\sin\phi)\cdot\frac{\sin^2(\frac{1}{2}\kappa\sin\phi)}{(\frac{1}{2}\kappa\sin\phi)^2} \quad \ldots\ldots(27),$$

the constant factor having been adjusted so as to make I_0 unity when $\phi = 0$.

Equations (23) and (26) complete the problem of the slit under the conditions postulated in the present paper, and they indicate to what extent the optical properties of the boundaries of the slit affect the theory of this celebrated problem. They further demonstrate rigidly the validity of Fresnel's formulae, which have always been deficient in this respect. It will be seen that the factor $\cos^2\phi/(1+\sin\phi)$ of (24) decreases as ϕ increases, while the factor $(1+\sin\phi)$ in (27) increases as ϕ increases.

If the incident wave is assumed to be a beam of circularly polarised light, that is a beam possessing an electric displacement in the wave-front parallel to the edges

of the slit combined with an electric displacement differing in phase by a quarter period at right angles to the edges, then from (24) and (27) the relative intensity in Fraunhofer's phenomena is

$$I_0 = \frac{\sin^2(\frac{1}{2}\kappa \sin\phi)}{(\frac{1}{2}\kappa \sin\phi)^2}\left\{\frac{\cos^2\phi}{1+\sin\phi} + 1 + \sin\phi\right\} = 2\,\frac{\sin^2(\frac{1}{2}\kappa \sin\phi)}{(\frac{1}{2}\kappa \sin\phi)^2} \qquad \text{.....(28).}$$

For general values of ϕ it is necessary to consider the complete expressions (23) and (26), and it may be here noted that the theory of the present paper is valid with little error even when r is as great as one-half of the width of the slit.

The foregoing theory shows that there are no minima where there is complete extinction as is the case in the first approximation. At the positions of the minima as given by the first approximation we conclude from (23) and (26) that, in the case of circularly polarised light, there is a distribution of intensity equal to

$$\frac{1}{\kappa^2}\left\{4\sin^2\tfrac{1}{2}\phi + \frac{1}{\cos^2\frac{1}{2}\phi}\right\} \qquad \text{.....(29).}$$

This expression increases with ϕ.

The optical properties of the boundaries of the slit have, therefore, the effect of propagating light into all parts of the space within the geometrical shadow.

Summary

For comparison we may set out the following results, the incident wave being in all cases denoted by

$$e^{i\kappa(y+ct)}.$$

In this case U_0/B of formula (4) corresponds to u of formulae (23) and (26), and

$$C_1 = \frac{1}{2\pi}\left(\frac{\pi}{2i\kappa\rho_0}\right)^{\frac{1}{2}} e^{-i\kappa(\rho_0 - ct)}.$$

I. When the wave-length is large in comparison with the width of the slit.

First approximation.

$$u/C_1 = 4i\kappa\cos^2\tfrac{1}{2}\phi.$$

Case A.

$$u/C_1 = -\frac{2\pi}{8}\kappa^2\cos\phi.$$

Case B.

$$u/C_1 = \frac{2\pi}{\gamma + \log(\frac{1}{2}i\kappa)},$$

γ being Euler's constant.

II. When the wave-length is small in comparison with the width of the slit.

First approximation.

$$u/C_1 = 2\cot\tfrac{1}{2}\phi\,(e^{i\kappa\sin\phi} - 1).$$

Case A.

$$u/C_1 = -\frac{\cot\phi}{(1+\sin\phi)^{\frac{1}{2}}}(e^{i\kappa\sin\phi} - 1) + 2\sin\tfrac{1}{2}\phi.$$

Case B.

$$u/C_1 = -\frac{(1+\sin\phi)^{\frac{1}{2}}}{\sin\phi}(e^{i\kappa\sin\phi} - 1) - \sec\tfrac{1}{2}\phi.$$

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PROCEEDINGS
AT THE
MEETINGS OF THE OPTICAL SOCIETY
1929–30

10th October, 1929

Meeting held at the Imperial College of Science.

Mr F. TWYMAN, President, in the Chair.

Elections (Membership): Dr J. J. HEDGES, and Dr J. J. BLANK.

The following paper was read:

"The Development of the Telescope, mostly after documents in the Court Collection (1675–1830)." (Third paper on the Court Collection.) By THOMAS H. COURT and MORITZ VON ROHR.

14th November, 1929

Meeting held at the Imperial College of Science.

Mr F. TWYMAN, President, in the Chair.

Election (Membership): Mr CLIFFORD BROWN.

Mr A. G. FREWIN, of Messrs The Emil Busch Optical Company, Limited, gave a short lecture on and demonstration of two instruments: "The Busch Optometer (Eye Refractor)," designed by Professor THORNER, and "A Glare-Free, Reflexless, Stereoscopic, Hand Ophthalmoscope for Oculists and Clinics," designed by Professor THORNER.

Mr E. F. FINCHAM gave a description and demonstration of his "Stereo-Fixation Device for Compimeter."

A Kinema film illustrating the Manufacture of Optical Glass was exhibited. The film was lent by the courtesy of Messrs Bausch and Lomb Optical Co., Ltd., and the Kinema Projector which was of the Home type was lent by the courtesy of Messrs Kodak, Ltd.

12th December, 1929

Meeting held at the Imperial College of Science.

Mr F. TWYMAN, President, in the Chair.

Elections (Membership): Mr W. E. HARDY, and Mr J. COLE.

The following paper was read:

"The Insensitivity and Personal Equation Errors of Optical Settings." By J. GUILD, A.R.C.S., F.Inst.P.

The following were demonstrated:

"A Cylindrical Mirror corrected to give freedom from aberration in the formation of an image of a distant line when the angle of incidence is 45°." By Messrs ADAM HILGER, Ltd. The mirror was described by Mr UNDERHILL.

"The Duochrome Refraction Test." By Mr CLIFFORD BROWN.

"The Maddox Cheiroscope." By Messrs C. W. DIXEY & SONS. The instrument was described by Mr S. B. ADAMS.

7th, 8th and 9th January, 1930

Twentieth Annual Exhibition of Electrical, Optical and other Physical Apparatus, held at the Imperial College of Science, jointly with the Physical Society.

The following discourses were given:

"Iridescent Colours in Nature from the Standpoint of Physical Optics." By The Right Hon. LORD RAYLEIGH, M.A., Sc.D., F.R.S.

"Gyro-Compasses for Gun-Fire Control." By Mr S. G. BROWN, M.I.E.E., F.Inst.P., F.R.S.

"Television, Present and Future." By SIR AMBROSE FLEMING, M.A., D.Sc., F.R.S.

16th January, 1930

Meeting held at the Imperial College of Science.

Mr J. GUILD, Vice-President, in the Chair.

Election (Membership): Professor H. LOWERY.

(Student Membership): Mr FRANCIS N. ARCHBOLD.

Messrs C. J. BECK and R. W. CHESHIRE were elected to act as Hon. Auditors of the Society's accounts for the year 1929.

The following papers were read:

"The Ross' Modification of the Hilger Interferometer for Testing large Optical Elements." By O. G. HAY, A.R.C.S., a Member of the Technical Staff of Messrs Ross, Ltd.

"The Reflection Factor of Magnesium Oxide." By Mr J. S. PRESTON, B.A., A.Inst.P., of the National Physical Laboratory.

Messrs Raphael's, Ltd., exhibited and described:

"A Scientype Occluding Glass for use in the Prevention of Squint," and "A Visual Test with a new system of Equi-photic Illumination."

13th February, 1930

Meeting held at the Imperial College of Science.

Mr F. TWYMAN, President, in the Chair.

Elections (Membership): Mr ERIC JOHN CRUNDALL, and Mr ROBERT DONALDSON.

The President delivered the Presidential Address entitled "Optics in Radio Transmission and other Fresh Fields."

13th March, 1930

Meeting held at the Imperial College of Science.

Mr F. TWYMAN, President, in the Chair.

Elections (Membership): Mr WILLIAM JOHN TENNANT, and Professor OSCAR ULRICH VONWILLER.

The following papers were read:

"Imagery around a Skew Ray." By Mr T. SMITH, M.A., of the National Physical Laboratory.

"The Purpose and Design of the New Equipment at the Royal Observatory, Edinburgh." By Professor R. A. SAMPSON, D.Sc., F.R.S.

"A description of the 36-inch Reflector at the Royal Observatory, Edinburgh." By Messrs SIR HOWARD GRUBB, PARSONS & Co., read by Mr Young.

"A description of the Stellar Spectrograph for the 36-inch Reflector at the Royal Observatory, Edinburgh." By Messrs ADAM HILGER, Ltd., read by Mr Dowell.

10th April, 1930

Annual General Meeting held at the Imperial College of Science.

Mr J. GUILD, Vice-President, in the Chair.

The Hon. Secretary presented the Annual Report* of the Council, which was approved.

The Hon. Treasurer presented his Report† and Balance Sheet‡, which were approved.

The Chairman read the list of persons nominated by the Council for office in the year 1929–30 and declared all these duly elected as no other nominations had been received:

President: Mr F. TWYMAN, F.R.S.
Vice-Presidents: Mr D. BAXANDALL, Mr E. F. FINCHAM, Mr J. GUILD, Professor A. F. C. POLLARD.
Hon. Secretary (Business): Mr W. B. COUTTS.
Hon. Secretary (Papers): Mr A. WHITWELL.
Hon. Treasurer: Major E. O. HENRICI.
Hon. Librarian: Mr J. H. SUTCLIFFE.
Editor of Transactions and Assistant Secretary: Dr J. J. HEDGES.

Council:

Already serving:	Elected April, 1930:
Mr W. B. BARKER.	Dr J. S. ANDERSON.
Mr T. CHAUNDY.	Dr W. M. HAMPTON.
Dr C. V. DRYSDALE.	Mr C. L. REDDING.
Mr S. A. EMERSON.	
Mr W. GAMBLE.	
Mr E. T. HANSON.	
Mr J. W. HASSELKUS.	
Professor A. O. RANKINE.	
Mr R. S. WHIPPLE.	

Votes of thanks to the retiring Officers and Members of Council, to the Hon. Auditors, to the Governors of the Imperial College, with Professor Rankine, for the use of the College premises for the meetings of the Society, and to Mr Banfield for his assistance at the meetings of the Society were unanimously passed.

10th April, 1930

Meeting held at the Imperial College of Science.

Mr J. GUILD, Vice-President, in the Chair.

Elections (Membership): Mr W. T. RICKETS.
(Fellowship): Mr D. P. BOATMAN, Mr H. F. BUCKLEY, M. ANDRÉ CALLIER, Mr F. C. DANNATT, Dr W. M. HAMPTON, Dr R. A. HOUSTOUN, Professor R. KINGSLAKE, Mrs KINGSLAKE, Mr J. LINN, Dr R. J. LYTHGOE, Mr Z. A. MERFIELD, Mr F. C. MUSGRAVE, Mr SEIJI NAKAMURA, Professor E. A. OWEN, Mr H. C. RANDS, Mr C. F. SMITH, and Mr J. S. WILSON.

The following papers were read:

"Charts for simple Two- and Three-Lens Problems." By Mr T. SMITH, M.A., F.Inst.P., of the National Physical Laboratory.

"The Lustre of Textile Fibres and a Method of Measurement." By Mr M. O. PELTON, B.Sc., of the British Cotton Industry Research Association.

"A Re-determination of the Mixture Curves of the Spectrum." By Mr W. D. WRIGHT, A.R.C.S., D.I.C., B.Sc.

* See p. ix. † See p. xiii. ‡ See pp. xiv, xv

8th May, 1930

Meeting held at the Imperial College of Science.

Mr F. TWYMAN, President, in the Chair.

Election (Student Membership): Mr DONALD JAMES WAITT.

The following paper was read:

"The Wedge Sector for Quantitative Spectrum Analysis." By Mr F. TWYMAN, F.R.S., and Dr F. SIMEON, B.Sc.

Mr E. F. FINCHAM demonstrated "A New Ophthalmoscope," and Mr RHEINBERG exhibited an interesting series of Kaleidoscopic patterns produced from photomicrographs, by HERR SCHMELIK, a Member of the Society.

19th June, 1930

Meeting held at the Imperial College of Science.

Dr J. S. ANDERSON in the Chair.

The following papers were read:

"The general form of the Smith-Helmholtz Equation." By T. SMITH, M.A., F.Inst.P.

"On the Diffraction of Light by a Slit." By E. T. HANSON, B.A.

"A Chromatic Method of making Optical Settings." By D. S. PERFECT, M.A., D.Phil., F.Inst.P.

Mr T. F. CONNOLLY, M.Sc., demonstrated two instruments:

"A Portable Standard Compass" and "A Self Checking Prismatic Compass."

THE OPTICAL SOCIETY

REPORT OF THE COUNCIL TO THE ANNUAL GENERAL MEETING OF MEMBERS, 10TH APRIL, 1930

THE COUNCIL of the Optical Society presents the following Report for the year ending 31st December, 1929.

During the year the Annual General Meeting and eight Ordinary Meetings of the Society were held at the Imperial College of Science and Technology by permission of the Governors and the late Professor Callendar.

The average attendance at the meetings was 33, which is about the same as for the year 1928. The Council is endeavouring to increase the interest of the meetings by including Demonstrations, whenever possible, in the ordinary business. It expresses the hope that Members will attend the meetings in larger numbers in future. Not only is the interest in the discussion of papers enhanced thereby, but a larger attendance is the best form of encouragement which can be given to those who present papers and demonstrate apparatus.

During the year ten papers were communicated, and ten demonstrations, in addition to those referred to below, were given at the Ordinary Meetings. Two important Historical Papers on the Court Collection, by Mr T. H. Court and Professor von Rohr, were presented at the May and October meetings, the papers being illustrated in each case from exhibits in the Court Collection. At the June meeting, by the kindness of Professor Rankine, a most important series of experiments and demonstrations, arranged for Members of the Society by Professor Pollard, Professor Martin and Mr Johnson, of the Technical Optics Department of the Imperial College, was inspected by some seventy Members and friends.

The Nineteenth Annual Exhibition of Electrical, Optical and other Physical Apparatus, under the auspices of the Optical and Physical Societies, was held at the Imperial College of Science and Technology on the 8th, 9th and 10th January. These Exhibitions are proving to be of increasing importance as the years go by, and this Exhibition broke all records as to the number of exhibitors and visitors.

During the Exhibition Discourses were delivered by Professor F. L. Hopwood, on "Experiments with High Frequency Sound Waves"; by Mr Conrad Beck, on "Lenses"; and by Mr A. J. Bull, on "Colour Problems in Photo-Engraving."

The Council records with regret the loss by death of one of its number, the late Mr O. Aves.

During the year Mr T. Martin resigned his office as Assistant Secretary on being appointed Secretary of the Royal Institution. The Council places on record its appreciation of the valuable services rendered to the Society by Mr Martin, and is sure that the Society will wish him all success in his new appointment.

Dr Anderson resigned from the office of Editor of the *Transactions*, which he had occupied for twelve years. The Council feels sure that the Members of the Society have appreciated the valuable services rendered to them by Dr Anderson during this long period, and in particular the very high standard to which the publications of the Society have been raised under the guidance of their Editor.

The Council has appointed Dr J. J. Hedges, Secretary of the Institute of Physics, to succeed Mr Martin as Assistant Secretary, and Dr Anderson as Editor of the *Transactions*.

As in former years, Members of the Society have been invited to attend the lectures on "Physics in Industry" by the Board of the Institute of Physics, and meetings of the Illuminating Engineering Society.

Mr Whipple has continued to represent the Society on the National Committee of Physics, and Dr Clay has represented the Society on the Board of the Institute of Physics.

Professor Pollard and Mr Sutcliffe have represented the Society on the Committee concerned with the establishment of a Joint Library at the offices of the Institute of Physics, the Physical Society and the Optical Society. As a step towards this, the Library of the Society, formerly housed at Clifford's Inn, is now at 1 Lowther Gardens. The thanks of the Society are due to the British Optical Association for so kindly housing the Society's books on its premises for so many years.

In September your Council received a request from the Executive Board of the Optical Convention (1926) that the Optical Society should take over its assets in trust. Your Council agreed to accept this trust under the following conditions:

(1) That payment should be made to the Guarantors as and when the accumulated funds amount to 10 per cent. of the Guarantee Fund.

(2) That should another Optical Convention of a similar character be held the remaining assets should be handed over to the Executive Committee of the new Board. The interpretation of "similar" to rest with the Council of the Optical Society.

(3) That should no Optical Convention of a similar nature be held within a period of 10 years from 1st January, 1930, the remaining assets should be dealt with at the sole discretion of the Council of the Optical Society.

These conditions were agreed to by the Guarantors of the Optical Convention (1926). The cash and stock of Publications have been taken over, and the *Proceedings of the Convention* are now on sale at the Society's Offices at 1 Lowther Gardens.

Towards the end of the year your President, acting on a suggestion by Mr Whipple, formed a Sub-Committee of the Council to meet a Sub-Committee of the Physical Society, with a view to organising a Joint Discussion on Photo-Electric Cells, to be held in June, 1930. A large number of papers have been promised, and the success of the discussion would appear to be assured.

LIBRARY

In addition to the regular journals the following publications have been received during 1929:

The Prescribing of Spectacles. Third edition. By ARCHIBALD STANLEY PERCIVAL, M.A., M.B., B.C.

Infra-red Analysis of Molecular Structure. By F. I. G. RAWLINS, M.Sc. and A. M. TAYLOR, M.A.

Applied Optics and Optical Design. Part I. By Prof. A. E. CONRADY.

Physiologische Optik. By Dr A. KÖNIG.

Forschungen zur Geschichte der Optik, 1 Band, 1 Heft. Edited by MORITZ VON ROHR.

The National Physical Laboratory—Report for the year 1928.

Bureau of Standards:

Research Paper No. 39. *Reflecting Power of Beryllium, Chromium, and several other metals*. By W. W. COBLENTZ and R. STAIR.

Research Paper No. 43. *Least Retinal Illumination by Spectral Light required to evoke the "Blue Arcs of the Retina."* By DEANE B. JUDD.

Miscellaneous Publication No. 86. *Tables of Spectral Energy Distribution and Luminosity for use in Computing Light Transmissions and Relative Brightnesses from Spectrophotometric Data*. By J. F. SKOGLAND.

Ophthalmological Optics (in Russian). By B. E. MOURASHKINSKY, S. J. MAISEL and G. A. MILK.

Proceedings of the Seventh International Congress of Photography, London, 1928. Edited by W. CLARK, T. SLATER PRICE and B. V. STORR.

Medical Research Council. Reports of the Committee upon the Physiology of Vision:

II. *Dark Adaptation* (*A Review of the Literature*). By DOROTHY ADAMS.

IV. *Experiments on Binocular Vision*. By N. M. S. LANGLANDS.

V. *The Adaptation of the Eye; its Relation to the Critical Frequency of Flicker*. By R. J. LYTHGOE and K. TANSLEY.

VI. *Some Experiments on Peripheral Vision*. By MYER SALAMAN.

Joseph Fraunhofers Leben, Leistungen und Wirksamkeit. Nach Quellen geschildert von MORITZ VON ROHR.

Professional Papers of the Air Survey Committee:

No. 5. *Calibration of Surveying Cameras*. By Captain M. HOTINE, R.E.

MEMBERSHIP

During the year twelve Members were elected. Two Corporate Members and seven Members resigned, four Members were removed and four died. There is, therefore, a decrease of five in the membership of the Society, leaving the number of Members on 1st January, 1930, as follows:

Ex-officio Fellow	1
Honorary Fellows	10
Fellows	121
Members	151
Corporate Members	9
Student Members	9
Total	301

The Council invites Members to co-operate with it in strengthening the position of the Society, by endeavouring to increase the membership. Application forms can be obtained from the Assistant Secretary, 1 Lowther Gardens, S.W. 7, at any time, or from the Secretaries at any Ordinary Meeting.

TREASURER'S REPORT FOR THE YEAR 1929

THE expenditure for the year has been normal, the increased payment to the Institute of Physics, mentioned in the last report, being offset by the increased receipt on behalf of the Exhibition expenses. The cost of printing and distributing the *Transactions* includes an estimated figure for the cost of Part 5, Vol. XXX, as difficulties in completing the manuscript of one paper have delayed the printing, and the cost is not yet known. The receipts from the sale of publications show an increase of £92 over those of last year. This is due largely to the sale of several complete sets of the *Transactions*, and it cannot be expected that this figure for sales will be maintained.

The net result for the year is a balance of income over expenditure of £144. 14*s*. 8½*d*., the most satisfactory result since the reduction in the subscription rates made in 1926. It would be still more satisfactory if the subscription income had increased as well as the income from sales of publications. It has been decided to expend part of the surplus on reprinting certain parts of the *Transactions* (Vol. XIV Part 3, Vol. XV, and Vol. XVI) which were out of print, at a cost of £108.

The balance sheet includes a new item, "Optical Convention, 1926, Trust Account, £122. 2*s*. 9*d*.," which represents the balance of cash remaining in the hands of the Executive Board of the Optical Convention, which Board has now been dissolved. This sum, together with the stock of publications, has been handed over to the Optical Society on condition that repayments should be made to the guarantors of the Convention as and when the money available amounts to 10 per cent. of the total guarantee fund. Should another Optical Convention of a similar character be held, the undistributed cash then remaining, and the stock of publications, will be handed over to the Executive Board of the new Convention. The receipt of the above sum accounts for the large cash balance shown in the balance sheet. (£100 of this has since been invested in 5 per cent. War Loan.) The investments remain as in the last two years; they are shown as "at cost" in the balance sheet. The market price on 31st December, 1929, showed a depreciation of £7, but the market price to-day is in excess of the cost price.

As usual, the assets shown in the balance sheet are all (with the possible exception of subscriptions due) liquid assets, and no credit has been taken for the value of the stock of publications, books or furniture, though this is considerable, and they are insured against loss by fire for the sum of £300.

E. O. HENRICI,
Hon. Treasurer.

10*th March*, 1930.

INCOME AND EXPENDITURE ACCOUNT FOR THE YEAR ENDING 31ST DECEMBER, 1929

Dr. EXPENDITURE

1928 £ s. d.		£ s. d.	£ s. d.
	To Printing and Stationery:		
	Printing and Distributing *Transactions*	444 3 2	
450 2 8	Office Printing and Stationery	28 10 11½	
37 8 9			472 14 1½
	Clerical Assistance:		
	Institute of Physics (includes administration of Exhibition)	125 0 0	
112 10 0	Editorial Expenses	48 14 2	
54 2 9	Other Expenses	28 18 7	
19 8 0			202 12 9
40 2 8½	Postage and Receipt Stamps		39 13 9½
30 13 10½	Expenses at Meetings		15 16 3½
7 19 4½	Sundry Expenses		5 0 11
12 10 3	Advertising		10 0 0
5 0 0	*Subscription to the Photographic Congress*		—
	Balance, being excess of Income over Expenditure, transferred to Accumulated Fund		144 14 8½
12 11 11½			
£782 10 4			£890 12 7

INCOME *Cr.*

1928 £ s. d.		£ s. d.	£ s. d.
	By Subscriptions:		
249 8 0	Fellows	246 14 3	
162 15 0	Members	158 11 0	
31 10 0	Corporate Members	31 10 0	
3 13 6	Students	3 3 0	
447 6 6		439 18 3	
	Less: Rebate to members of the Institute of Physics 11 0 4		
	Subscriptions written off 7 7 0		
15 10 2		18 7 4	
431 16 4			421 10 11
29 8 6	Advertisements in *Transactions*, *less* expenses		42 15 6
240 0 10	Sale of Publications		332 16 1
35 12 2	Interest and Dividends		37 7 10
45 12 6	Physical Society—Exhibition Account		56 2 3
£782 10 4			£890 12 7

BALANCE SHEET AT 31ST DECEMBER, 1929

LIABILITIES

	£ s. d.	£ s. d.
Sundry Creditors		218 9 11
Accumulated Funds	501 17 8½	
Add Balance of Income and Expenditure Account	144 14 8½	
		646 12 5
Life Composition Account:		
Balance at 31st December, 1928		94 10 0
Subscriptions Received in Advance		10 9 5
Optical Convention, 1926, Trust Account		122 2 9
		£1092 4 6

ASSETS

	£ s. d.	£ s. d.
Sundry Debtors		58 7 7
Subscriptions Due		9 9 0
Investment:		
£650 5% War Loan (at cost)		657 6 3
Cash at Bank: Deposit Account	100 0 0	
Current Account	253 18 4	
Cash in Hand	13 3 4	
		367 1 8
		£1092 4 6

E. O. HENRICI,
Hon. Treasurer.

We have audited the above Balance Sheet and have obtained all the information and explanations we have required. We have verified the Bank Balances and the Investment. In our opinion such Balance Sheet is properly drawn up so as to exhibit a true and correct view of the state of the Society's affairs according to the best of our information and the explanations given to us and as shown by the books of the Society.

KNOX, CROPPER & CO.
Chartered Accountants.

Spencer House,
South Place, E.C. 2.
18th March, 1930.

Examined and approved on behalf of the Society.

26th March, 1930.

C. J. BECK,
R. W. CHESHIRE,
Hon. Auditors.